

Row-Reducing the Quantum Determinant and Dieudonné Determinant

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Abstract

We prove that row reducing a quantum matrix yields another quantum matrix for the same parameter q . This means that the elements of the new matrix satisfy the same relations as those of the original quantum matrix ring $M_q(n)$. As a corollary, we can prove that the image of the quantum determinant in the abelianization of the total ring of quotients of $M_q(n)$ is equal to the Dieudonné determinant of the quantum matrix. A similar result is proved for the multiparameter quantum determinant.

1 Introduction

The representation theory of the quantum general linear group is similar to that of the commutative coordinate ring of the general linear group.

In this paper we show that row-reducing, an important method in linear algebra, does also work for quantum matrices and the quantum determinant. More precisely, in Theorem 3, we show that the matrix obtained from the original quantum matrix Z by row reducing to clear all the elements, but one, in the first column, yields a matrix that is again a quantum matrix, such that, the elements of the new matrix satisfy the same relations as those of the original quantum matrix ring $M_q(n)$. In Theorem 4 we prove that the quantum determinant of the new (row-reduced) matrix, is equal to the quantum determinant of the original quantum matrix.

We work over an algebraically closed field k . The ring of quantum matrices $M_q(n)$ is an iterated Ore extension of k , hence it is an integral domain and it has a total ring of left (and right) quotients D . We show in Corollary to Theorem 4 that the image of the quantum determinant of the quantum matrix Z in D^{*ab} is equal to the Dieudonné determinant of the matrix Z . This can also be established by using the remark following 3.1 and 3.3 in the paper [GR1] by Gelfand and Retakh. The proofs of the above mentioned formulas are not provided, therefore, we shall give a detailed account of our proof. On the other hand our approach is quite different. It is based on row-reducing a quantum matrix and on the fact that the quantum determinant is invariant under row reducing over D . Our proof in contrast to the approach in [GR1] and [GR2] has to do with a Grassmannian algebra.

Although both are called determinants, the Dieudonné determinant and the quantum determinant each represent two quite different approaches for the notion of a noncommutative

determinant. Basically, the Dieudonné determinant comes from row reducing, while the quantum determinant comes from a Grassmannian algebra. The major difficulty is caused by the fact that the quantum determinant is not, in general, multiplicative.

The Dieudonné determinant was defined in [Di] for an arbitrary square matrix with coefficients in a skew field; its values are cosets in D^{*ab} . While one can provide a well-defined element in D^{*ab} , we do not get a nice formula for it. To calculate the Dieudonné determinant we use a procedure, similar to the one of producing the row-reduced echelon form of a matrix.

The quantum determinant cannot be defined for an arbitrary matrix but only for a quantum matrix. For a quantum matrix we may consider a quantum Grassmannian plane, which is a Frobenius quadratic algebra and has an associated quantum determinant, see [Ma], Ch.8. The value of the quantum determinant is given by an elegant polynomial formula. It also has a nice formula for row or column expansion. In fact, the construction of the quantum determinant is possible because one has a categorical braiding which can be used to define the quantum Grassmannian algebra. Therefore, the quantum determinant corresponds to a setting which entails information not ordinarily present when one just looks for a noncommutative determinant of a square matrix over an arbitrary skew field. This explains why the quantum determinant has properties that do not persist for the more general Dieudonné determinant.

For $n = 2$ the expression of the quantum determinant is $\det_q(Z) = ad - q^{-1}bc = ad - aca^{-1}b$, therefore its projection in D^{*ab} , is equal to the Dieudonné determinant. A direct calculation of the Dieudonné determinant, for large n , does not look too nice anymore. We establish eventually the similar fact for an arbitrary n by using an inductive argument.

The first two sections of this paper recall basic well known facts about the Dieudonné determinant and the quantum determinant. Then we establish the Theorems 3 and 4 and as a corollary the fact that the image of $\det_q(Z)$ in D^{*ab} equals the Dieudonné determinant for the quantum matrix for any n . Finally we prove that a similar result holds for the multiparameter quantum determinant as defined in the paper by Artin, Schelter and Tate [AST].

2 The Dieudonné determinant.

Let D be a skew field and denote by π the canonical surjection $D \longrightarrow D^{*ab} = D^*/[D^*, D^*]$. Recall some notations and definitions from [Dx]. Let $M_n(D)$ be the matrix units in the ring of square matrices of dimension n over a skew field D , and denote by $e_{i,j}$ the matrix units in $M_n(D)$.

As usual we denote the transvections by $\tau_{i,j}(t) = I + te_{i,j}$, and we let $d_i(u) = I + (u - 1)e_{i,i}$. For an arbitrary permutation $\sigma \in S_n$ we associate the permutation matrix $P(\sigma) = (\delta_{i,\sigma j})$ where δ denotes the Kronecker symbol.

Multiplication on the left (right) by these matrices will perform the usual elementary row (column) transformations.

Recall now that a matrix $A \in M_n(D)$ is right invertible if and only if it is left invertible, and

this happens if and only if the rows (columns) of A are left (right) linearly independent. We denote the set of invertible matrices in $M_n(D)$ by $GL_n(D)$.

The original design for the Dieudonné determinant was to be a multiplicative mapping $\det_D : M_n(D) \longrightarrow D^{*ab}$, equal to 1 for any transvection $\tau_{i,j}(t)$. The price to pay for such a general definition is that its values are not in D but are cosets in $D^{*ab} = D^*/[D^*, D^*]$. It is difficult to handle this determinant because its values are not polynomials but rational functions in the coefficients and it hasn't any nice formula for row or column expansion. There is a substitute for the row expansion given by Theorem 4.5, Ch.IV in [Ar], but it is too weak to be useful.

We need the following result, see [Dx], Theorems 1 and 2 in Ch.19.

Theorem 1 (Bruhat normal form) *Let D be a skew field and $A \in M_n(D)$ a matrix having a right inverse. Then there exists a decomposition*

$$A = TUP(\sigma)V \tag{1}$$

$$\text{where } T = \begin{pmatrix} 1 & * & * \\ & \ddots & * \\ 0 & & 1 \end{pmatrix}, \quad U = \text{diag}(u_1, \dots, u_n), \quad V = \begin{pmatrix} 1 & & 0 \\ * & \ddots & \\ * & * & 1 \end{pmatrix}$$

σ a permutation, $P(\sigma)$ the permutation matrix corresponding to σ and U and σ are unique with these properties.

Now we may introduce the following:

Definition *Let $A \in M_n(D)$, and let $A = TUP(\sigma)$, $U = \text{diag}(u_1, \dots, u_n)$, with u_1, \dots, u_n and σ uniquely defined by the Bruhat normal form of A . Define $\delta\epsilon\tau : M_n(D) \longrightarrow D$, by:*

$$\begin{aligned} \delta\epsilon\tau(A) &= 0 \text{ if } A \text{ is not invertible and} \\ \delta\epsilon\tau(A) &= \text{sign}(\sigma) u_1 \dots u_n \text{ if } A \in GL_n(D). \end{aligned}$$

Definition *Let $\pi : D^* \longrightarrow D^{*ab} = D^*/[D^*, D^*]$ be the canonical projection. The Dieudonné determinant of A is:*

$$\det_D(A) = \pi [\delta\epsilon\tau(A)].$$

Occasionally we denote the Dieudonné determinant by $|A|_D$.

A direct consequence of this definition is that the Dieudonné determinant has a number of remarkable properties which are similar to properties of a determinant in the commutative case:

- Proposition 1** 1) $\det_D(\tau_{i,j}(t)) = 1$ for any transvection $\tau_{i,j}(t) = I + te_{i,j}$
 2) $\det_D(P(\sigma)) = \text{sign}(\sigma)$ for any permutation matrix σ .
 3) The Dieudonné determinant is multiplicative: $\det_D(A \cdot B) = \det_D(A) \cdot \det_D(B)$

For the proofs see the original paper [Di], [Dx] Ch.19, 20 or [Ar] Ch.IV.

The following two examples may help explain some of the reasons for taking cosets modulo commutators in D in defining the Dieudonné determinant:

Example 1 Let $U = \begin{pmatrix} u_1 & & 0 \\ & \ddots & \\ 0 & & u_n \end{pmatrix}$ and $V = \begin{pmatrix} v_1 & & 0 \\ & \ddots & \\ 0 & & v_n \end{pmatrix}$

Then $\delta\epsilon\tau(U) = u_1 \dots u_n$, $\delta\epsilon\tau(V) = v_1 \dots v_n$ and $\delta\epsilon\tau(UV) = u_1 v_1 \dots u_n v_n$.

This shows that $\delta\epsilon\tau$ is not multiplicative while $\pi(\delta\epsilon\tau(UV)) = \pi(\delta\epsilon\tau(U)) \cdot \pi(\delta\epsilon\tau(V))$.

Hence multiplicativity will not hold unless we use the projection by π .

Example 2 Let $U = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$ and $V = \begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix}$. Then V can be obtained from U by elementary row transformations, so that one can write $V = TU$ with T a product of transvections. Here $\delta\epsilon\tau(U) = uv$ and $\delta\epsilon\tau(V) = vu$; they are not necessarily equal while $\pi(\delta\epsilon\tau(U)) = \pi(\delta\epsilon\tau(V))$

We calculate now the Dieudonné determinant for the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ when $a \neq 0$.

Example 3 (The Dieudonné determinant for a 2 by 2 matrix)

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ ca^{-1} & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d - ca^{-1}b \end{pmatrix} = \\ &= \begin{pmatrix} 1 & 0 \\ ca^{-1} & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & d - ca^{-1}b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix} \end{aligned}$$

therefore

$$\left| \begin{array}{cc} a & b \\ c & d \end{array} \right|_D = \pi(ad - aca^{-1}b). \quad (2)$$

Here we denoted the Dieudonné determinant by $|A|_D$.

Remark 1 Not all the properties for the commutative determinant will hold. For instance

- there is no row expansion formula.
- the Dieudonné determinant of the transpose of a matrix is not equal to the original determinant:

For a noncommutative D and two elements a, b in D such that $ab \neq ba$, the rows of the matrix $A = \begin{pmatrix} 1 & a \\ b & ab \end{pmatrix}$ are left linearly independent hence the Dieudonné determinant is non-zero. The transpose $A^t = \begin{pmatrix} 1 & b \\ a & ab \end{pmatrix}$ has proportional rows. Hence $\det_D(A) \neq \det_D(A^t) = 0$ see [Ar], Ch.IV, example following Thm.4.4.

We shall need the following result established in the original paper by Dieudonné [Di], 1⁰ in the proof of Thm.1. In fact Dieudonné originally defined his noncommutative determinant by establishing first the following

Theorem 2 (Dieudonné) *Let $A = (a_{ij})$ and select a non-zero element in the first column, $a_{i1} \neq 0$. Let $A' = (a'_{ij})$ be the matrix obtained from A by using elementary row operations to clear all elements in the first column except a_{i1} i.e. subtract the i -th row multiplied on the left by $a_{j1}a_{i1}^{-1}$ from the j -th row. Denote by A_{ij} the $(n-1) \times (n-1)$ matrix obtained by deleting the first column and the i -th row from the original A . Then for any i such that $a_{i1} \neq 0$ the product $\pi(a_{i1}) \cdot \det_D(A_{i1})$ is the same, and it is equal to the Dieudonné determinant $\det_D(A)$ of A , i.e. $\det(A) = \pi(a_{i1}) \cdot \det_D(A_{i1})$.*

3 The Quantum determinant

The notion of quantum determinant was introduced in the theory of quantum groups, see [Dr], [Ma], [FRT] and [Gu]. The quantum determinant is a central element in the bialgebra of quantum matrices, and we shall see that it can be defined by a nice formula. It also has nice formulas for row or column expansion. Being a group-like element in the bialgebra of quantum matrices, it has a weak multiplicative property when applied to quantum matrices, such that their components commute pairwise. In that case the product of the two quantum matrices is again a point of $M_q(n)$. Keep also in mind that the quantum determinant can be defined for quantum matrices that are not square matrices (see Ch. 8, example 7, [Ma]).

Definition *Let A be a graded k -algebra $A = \bigoplus_{i \geq 0} A_i$ such that $A_0 = k$, $\dim_k(A_1) < \infty$, A is generated by A_1 as an algebra and the ideal of relations is generated by a set of homogeneous generators of degree 2.*

Two very important examples of quadratic algebras are:

- The *quantum plane*: $A = k \langle x, y \rangle / (yx - qxy)$
- The *quantum Grassmannian*: $B = k \langle \xi, \eta \rangle / (\xi^2, \eta^2, \xi\eta - q\eta\xi)$.

Another important example of the notion of quadratic algebra introduced by Manin is the quantum matrix bialgebra $M_q(2)$. Let k be a field and $q \in k^*$.

- For $n = 2$ the *quantum matrix bialgebra* is given by:
 $M_q(2) = k \langle a, b, c, d \rangle$ subject to the following relations:
 $ab = q^{-1}ba, \quad cd = q^{-1}dc$
 $ac = q^{-1}ca, \quad bd = q^{-1}db$
 $bc = cb, \quad ad - da = (q^{-1} - q)bc.$

The comultiple is defined as the dual of matrix multiplication namely:

$$\Delta(a) = a \otimes a + b \otimes c$$

$$\Delta(b) = a \otimes b + b \otimes d$$

$$\Delta(c) = c \otimes a + d \otimes c$$

$$\Delta(d) = c \otimes b + d \otimes d$$

or in short:

$$\Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The counit is determined by $\epsilon(a) = \epsilon(d) = 1, \epsilon(b) = \epsilon(c) = 0$.

Let A be an algebra over k and let $\varphi : A \longrightarrow A$ be a k -endomorphism. We call a linear endomorphism δ a φ -derivation of A if $\delta(ab) = \varphi(a)\delta(b) + a\delta(b)$, for all $a, b \in A$. For most of the following results one usually asks φ to be injective, in our case it will be so granted that φ is a k -automorphism.

Definition If $A \subseteq B$, then we say B is a left Ore extension or skew polynomial extension of A and write $B = A[X, \varphi, \delta]$ if B is a ring of noncommutative polynomials in the variable X with coefficients in A and the commutation relations between coefficients and variables are given by: $Xa = \varphi(a)X + \delta(a)$, for all $a \in A$.

We have then that $M_q(2)$ is an iterated Ore extension of k , as shown by the following tower of Ore extensions (see [Ka] Thm IV.4.10):

$$k \hookrightarrow k[a, id, 0] \hookrightarrow k[a][b, \varphi_1, 0] \hookrightarrow k[a, b][c, \varphi_2, 0] \hookrightarrow M_q(2) = k[a, b, c, d] = k[a, b, c][d, \varphi_3, \delta]$$

$\varphi_1 : k[a] \rightarrow k[a]$ is a k -morphism such that $\varphi_1(a) = qa$,

$\varphi_2 : k[a, b] \rightarrow k[a, b]$ is a k -morphism such that $\varphi_2(a) = qa, \varphi_2(b) = b$

$\varphi_3 : k[a, b, c] \rightarrow k[a, b, c]$ is a k -morphism with $\varphi_3(a) = a, \varphi_3(b) = qb, \varphi_3(c) = qc$

$\delta : k[a, b, c] \rightarrow k[a, b, c]$ a φ_3 -derivation defined by: $\delta(a) = (q - q^{-1})bc, \delta(b) = \delta(c) = 0$.

When A is a domain and B is an Ore extension then B is also a domain. For this reason $M_q(2)$ is a domain. When A is left noetherian, then the Ore extension B is also left noetherian ([Ka] Proposition I.8.3), therefore the left Ore condition is satisfied (see [Co] proposition 1.3.2) for the multiplicative system of non-zero elements and one can construct a total field of fractions of $M_q(2)$, that will be denoted in the sequel by D .

Let $E = k \langle a, b, c, d \rangle$ with a, b, c, d subject to the relations in $M_q(2)$ and $R = E \otimes_k B$. It means R is generated over k by a, b, c, d and by ξ and η (subject to the relations in B), and any of a, b, c, d commutes with any of ξ and η . Let us denote :

$$x' = ax + by, \quad y' = cx + dy \quad \text{and} \quad \xi' = a\xi + b\eta, \quad \eta' = c\xi + d\eta$$

We say x', y' is a point of the quantum plane if $y'x' = qx'y'$ and ξ', η' is a point of the quantum Grassmannian if $\xi'^2 = 0, \eta'^2 = 0$, and $\xi'\eta' = q\eta'\xi'$.

Kobyzev noticed, see [Ma], Ch1, that a, b, c and d satisfy the relations defining $M_q(2)$, if and only if x', y' is a point of the quantum plane and ξ', η' is a point of the quantum Grassmannian.

The quantum determinant for $n = 2$ is defined to be: $det_q = ad - q^{-1}bc$, this is just the unique element $det_q(Z)$ in $M_q(2)$ such that $\xi'\eta' = det_q(Z) \cdot \xi\eta$ in R .

Using now the commutation relations in $M_q(2)$ we get $det_q(Z) = ad - aca^{-1}b$.

If we think of the quantum determinant as being an element in D associated to the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then the image of $det_q(Z)$ in D^{*ab} is equal to the value of the Dieudonné determinant for this matrix.

On the other side direct calculation shows that the quantum determinant $det_q(Z)$ is a group-like element and commutes with a, b, c, d . Therefore det_q is central in $M_q(2)$.

For an arbitrary n define

- the *quantum plane*: $A = k \langle x_1, \dots, x_n \rangle$ subject to: $x_j x_i - q x_i x_j = 0$ for all $i < j$
- the *quantum Grassmannian plane*: $B = k \langle \xi_1, \dots, \xi_n \rangle$ subject to: $\xi_i^2 = 0$ for all i , and $\xi_i \xi_j - q^{-1} \xi_j \xi_i = 0$ for all $i < j$
- the bialgebra of *quantum matrices* to be the noncommutative ring

$M_q(n) = k \langle z_{11}, z_{12}, \dots, z_{nn} \rangle$ subject to the following relations:

$$\mathcal{Rel}_q \left\{ \begin{array}{l} \text{Row relations: } z_{il} z_{ik} = q z_{ik} z_{il} \text{ for all } k < l \\ \text{Column relations: } z_{jk} z_{ik} = q z_{ik} z_{jk} \text{ for all } i < j \\ \text{Secondary diagonal relations: } z_{il} z_{jk} = z_{jk} z_{il} \text{ for all } i < j, k < l \\ \text{Main diagonal relations: } z_{jl} z_{ik} - z_{ik} z_{jl} = (q^{-1} - q) z_{il} z_{jk} \text{ for all } i < j, k < l. \end{array} \right.$$

We denote by Z the matrix (z_{ij}) such that z_{ij} satisfy the relations \mathcal{Rel}_q of $M_q(n)$.

The comultiplication on $M_q(n)$ is dual to matrix multiplication, i.e., $\Delta(z_{ij}) = \sum_k z_{ik} \otimes z_{kj}$ and the counit is defined by $\epsilon(z_{ij}) = \delta_{ij}$, both extended multiplicatively to all $M_q(n)$. Then $M_q(n)$ is a bialgebra.

Notice that one way of looking at these relations is asking whether for any selection of 2 rows and 2 columns the 4 elements at the intersection of the selected rows and columns satisfy the relations demanded before for a, b, c, d in the case $n = 2$.

Denote again $E = k \langle z_{ij} \rangle$, $R = E \otimes_k B$, the observation made by Kobyzev still holds:

Proposition 2 *Let*

$$x'_i = \sum_j z_{ij} x_j \quad \text{and} \quad \xi'_i = \sum_j z_{ij} \xi_j.$$

Then z_{ij} satisfy the defining equations of $M_q(n)$ if and only if x'_i (respectively ξ'_i) satisfy the defining relations of the quantum plane (respectively the quantum Grassmannian plane).

Now we follow Manin [Ma], Ch8 to define the quantum determinant

Definition A quadratic algebra A is called a Frobenius algebra of dimension d if $\dim(A_d) = 1$, $A_i = 0$ for $i > d$ and for all j , $0 \leq j \leq d$ the multiplication map $m : A_j \otimes A_{d-j} \longrightarrow A_d$ is a perfect duality. If in addition $\dim A = \binom{d}{i}$, call A a quantum Grassmannian algebra.

Definition Let A be a Frobenius algebra and let E be a bialgebra coacting on A by $\delta : A \longrightarrow E \otimes A$ and such that $\delta(A_1) \subseteq E \otimes A_1$. Then by induction, for any j one has $\delta(A_j) \subseteq E \otimes A_j$, and in particular since $\dim(A_d) = 1$, there is an element called the quantum determinant of the coaction $D = DET(\delta) \in E$ such that for any $a \in A_d$ one has:

$$\delta(a) = DET(\delta) \otimes a$$

An immediate consequence of coassociativity is that $DET(\delta)$ is a group-like element:

$$\Delta(DET(\delta)) = DET(\delta) \otimes DET(\delta).$$

It is easy to see now, cf. [Ma] Ch8 Example 6 that:

Proposition 3 The quantum Grassmannian plane is a Frobenius quadratic algebra, in fact even a quantum Grassmannian algebra. The quantum determinant in this case is given by the formula:

$$|Z|_q = \det_q(Z) = \sum_{\sigma \in S_n} (-q^{-1})^{l(\sigma)} z_{1\sigma(1)} z_{2\sigma(2)} \cdots z_{n\sigma(n)} \quad (3)$$

where $l(\sigma)$ is the length of σ , equal to the number of inversions in σ .

Proposition 4 Let $E = k \langle z_{ij} \rangle$, $R = E \otimes_k B$, and $\xi'_i = \sum_j z_{ij} \xi_j$. Then in R we have:

$$\xi'_1 \xi'_2 \cdots \xi'_n = \det_q(Z) \xi_1 \xi_2 \cdots \xi_n$$

An immediate consequence of the definition of the quantum determinant is the Laplace expansion formulas for rows and columns expansion. Denote by Z_{ji} the quantum determinant of the $(n-1)$ by $(n-1)$ matrix obtained by removing the j -th row and the i -th column of Z . $\tilde{Z} = ((-q)^{j-i} Z_{ji})$ is called the q -cofactor matrix. The following is proved in [FRT] Thm 4, see also [Tk] Proposition 2.3 and [PW].

Proposition 5 If Z is a quantum matrix then $Z\tilde{Z} = \tilde{Z}Z = \det_q(Z)I$. Consequently $\det_q(Z)$ is central and the following row expansion formulas hold:

$$\det_q(Z) = \sum_j (-q)^{i-j} z_{ij} Z_{ij} = \sum_i (-q)^{i-j} z_{ij} Z_{ij}.$$

There is an algebra automorphism $\tau : M_q(n) \longrightarrow M_q(n)$, called the transposition and defined by extending multiplicatively from $\tau(z_{ij}) = z_{ji}$. It is known that the quantum determinant is invariant under τ .

We remark now that exactly as in the case $n = 2$ shown before the ring of quantum matrices $M_q(n) = k \langle z_{11}, z_{12}, \dots, z_{nn} \rangle$ is an iterated Ore extension of k , therefore we can establish inductively that it is a left and right noetherian domain. Hence $M_q(n)$ is a domain and satisfies the Ore condition for the multiplicative system of non-zero elements. From now on we shall denote by D the skew field that is the total ring of fractions of $M_q(n) = k \langle z_{11}, z_{12}, \dots, z_{nn} \rangle$.

Definition *Let R be a k -algebra. We call an R -point of $M_q(n)$, (respectively of the quantum plane, or quantum Grassmannian plane) any k -algebra morphism in $\text{Alg}_k(M_q(n), R)$ (respectively in $\text{Alg}_k(A, R)$ or $\text{Alg}_k(B, R)$)*

Any k -algebra morphism $\Phi : M_q(n) \longrightarrow R$ is uniquely determined by a matrix with entries in R given by $r = (r_{ij}) = \Phi(z_{ij})$ such that the elements r_{ij} satisfy the relations $\mathcal{R}el_q$ for quantum matrices. Given such a matrix we shall denote by Φ_r the R point of $M_q(n)$ uniquely determined by r , and we shall say that r is a quantum matrix.

Definition *The quantum determinant evaluated at an R point r of $M_q(n)$ is*

$$|r|_q = \Phi_r(\det_q(X)) = \sum_{\sigma \in S_n} (-q^{-1})^{l(\sigma)} r_{1\sigma(1)} r_{2\sigma(2)} \cdots r_{n\sigma(n)}.$$

4 The main results

Let us now define a new matrix $Z' = (z'_{ij})$ by :

$$z'_{1j} = z_{1j} \text{ for any } j, z'_{i1} = 0 \text{ for } i = 2, \dots, n \quad (4)$$

$$z'_{ij} = z_{ij} - z_{i1}z_{11}^{-1}z_{1j} \text{ for } i = 2, \dots, n \text{ and } j = 2, \dots, n. \quad (5)$$

Note that Z' is obtained from Z by clearing up all the positions in the first column except the first one by use of elementary row transformations.

Let Z'' be the $(n-1) \times (n-1)$ matrix obtained by deleting the first row and the first column of Z' .

Theorem 3 (Row-reducing a quantum matrix) *The matrix Z' is a quantum matrix, its elements satisfy the defining relations $\mathcal{R}el_q$ of $M_q(n)$.*

Proof. For a selection of first row with any other row and any two columns different from the first, we may check directly that the relations are satisfied. An observation that makes

calculations easier is that $z'_{ij} = z_{11}^{-1} \begin{vmatrix} z_{11} & z_{1i} \\ z_{j1} & z_{ji} \end{vmatrix}_q$, now for $a = z_{11}, b = z_{1i}, c = z_{j1}, d = z_{ji}$ one may use the fact that the quantum determinant $ad - q^{-1}bc$ commutes with a, b, c, d .

For the other selections direct calculations will prove that if $A = z'_{ik}, B = z'_{il}, C = z'_{jk}, D = z'_{jl}$ and $i < j, k < l$ then A, B, C and D satisfy the relations for quantum matrices. of general i, j, k, l none of them = 1) works in exactly the same way. All the proofs work in a similar way: we express all monomials in terms of a Poincaré-Birkhoff-Witt basis of monomials in z'_{ij} s with indices in increasing lexicographic order by using the commutation relations and look for cancellations.

Lemma 1 $BC = CB$

Proof. We express $BC - CB$ in terms of a basis of monomials in z'_{ij} s with indices in increasing lexicographic order using the commutation relations

$$\begin{aligned}
BC - CB &= (z_{il} - z_{i1}z_{11}^{-1}z_{1l})(z_{jk} - z_{j1}z_{11}^{-1}z_{1k}) - (z_{jk} - z_{j1}z_{11}^{-1}z_{1k})(z_{il} - z_{i1}z_{11}^{-1}z_{1l}) = \\
&\underbrace{z_{il}z_{jk}}_{\boxed{1}} - \underbrace{z_{il}z_{j1}z_{11}^{-1}z_{1k}}_{\boxed{2}} - \underbrace{z_{i1}z_{11}^{-1}z_{1l}z_{jk}}_{\boxed{3}} + \underbrace{z_{i1}z_{11}^{-1}z_{1l}z_{j1}z_{11}^{-1}z_{1k}}_{\boxed{4}} + \\
&- \underbrace{z_{jk}z_{il}}_{\boxed{1'}} + \underbrace{z_{j1}z_{11}^{-1}z_{1k}z_{il}}_{\boxed{2'}} + \underbrace{z_{jk}z_{i1}z_{11}^{-1}z_{1l}}_{\boxed{3'}} - \underbrace{z_{j1}z_{11}^{-1}z_{1k}z_{i1}z_{11}^{-1}z_{1l}}_{\boxed{4'}} \\
\boxed{2} &= q^{-1}z_{11}^{-1}z_{1k}z_{il}z_{j1} \\
\boxed{3} &= q^{-1}z_{11}^{-1}z_{1l}z_{i1}z_{jk} \\
\boxed{2} &= q^{-1}(z_{il}z_{11}^{-1})z_{1k}z_{j1} = (\text{ use now the formula } da^{-1} - a^{-1}d = (q^{-1} - q)a^{-1}bca^{-1} = \\
&= (q^{-1} - q)q^{-2}a^{-2}bc) \\
&= q^{-1}z_{11}^{-1}(z_{il}z_{1k})z_{j1} + (q^{-1} - q)q^{-2}z_{11}^{-2}z_{1k}z_{1l}z_{i1}z_{j1} = (\text{use now } ad - da = (q^{-1} - q)bc) \\
&= q^{-1}z_{11}^{-1}z_{1k}z_{il}z_{j1} - (q^{-1} - q)q^{-1}z_{11}^{-1}z_{1l}z_{ik}z_{j1} + (q^{-1} - q)q^{-2}z_{11}^{-2}z_{1k}z_{1l}z_{i1}z_{j1} \\
\boxed{3} &= q^{-1}(z_{jk}z_{11}^{-1})z_{1l}z_{i1} = q^{-1}z_{11}^{-1}z_{1l}(z_{jk}z_{i1}) + (q^{-1} - q)q^{-2}z_{11}^{-2}z_{1k}z_{1l}z_{i1}z_{j1} = \\
&= q^{-1}z_{11}^{-1}z_{1l}z_{i1}z_{jk} - (q^{-1} - q)q^{-1}z_{11}^{-1}z_{1l}z_{ik}z_{j1} + (q^{-1} - q)q^{-2}z_{11}^{-2}z_{1k}z_{1l}z_{i1}z_{j1}
\end{aligned}$$

Now one can see that all like terms cancel. □

Lemma 2 $BA = qAB$

Proof.

$$\begin{aligned}
BA - qAB &= \\
&= (z_{il} - z_{i1}z_{11}^{-1}z_{1l})(z_{ik} - z_{i1}z_{11}^{-1}z_{1k}) - q(z_{ik} - z_{i1}z_{11}^{-1}z_{1k})(z_{il} - z_{i1}z_{11}^{-1}z_{1l}) = \\
&\underbrace{z_{il}z_{ik}}_{\boxed{1}} - \underbrace{z_{il}z_{i1}z_{11}^{-1}z_{1k}}_{\boxed{2}} - \underbrace{z_{i1}z_{11}^{-1}z_{1l}z_{ik}}_{\boxed{3}} + \underbrace{z_{i1}z_{11}^{-1}z_{1l}z_{i1}z_{11}^{-1}z_{1k}}_{\boxed{4}} + \\
&- \underbrace{qz_{ik}z_{il}}_{\boxed{1'}} + \underbrace{qz_{i1}z_{11}^{-1}z_{1k}z_{il}}_{\boxed{2'}} + \underbrace{qz_{ik}z_{i1}z_{11}^{-1}z_{1l}}_{\boxed{3'}} - \underbrace{qz_{i1}z_{11}^{-1}z_{1k}z_{i1}z_{11}^{-1}z_{1l}}_{\boxed{4'}}
\end{aligned}$$

Now $\boxed{1} = \boxed{1'}$ and $\boxed{4} = \boxed{4'}$ and:

$$\begin{aligned}
\boxed{2} &= z_{11}^{-1} z_{1k} z_{i1} z_{il} \\
\boxed{3} &= q^{-1} z_{11}^{-1} z_{1l} z_{i1} z_{ik} \\
\boxed{2} &= q^{-1} (z_{il} z_{11}^{-1}) z_{i1} z_{1k} = q^{-1} z_{11}^{-1} (z_{il} z_{1k}) z_{i1} - (q^{-1} - q) q^{-2} z_{11}^{-2} z_{1l} z_{i1} z_{1k} z_{i1} = \\
&= q^{-1} z_{11}^{-1} z_{1k} z_{il} z_{i1} - (q^{-1} - q) z_{11}^{-1} z_{1l} z_{i1} z_{ik} - (q^{-1} - q) q^{-2} z_{11}^{-2} z_{1k} z_{1l} z_{i1}^2 \\
\boxed{3} &= (z_{ik} z_{11}^{-1}) z_{i1} z_{1l} = q z_{11}^{-1} z_{1l} z_{i1} z_{ik} + (q^{-1} - q) q^{-2} z_{11}^{-2} z_{1k} z_{1l} z_{i1}^2
\end{aligned}$$

The cancellations are now clear. \square

Lemma 3 $CA = qAC$

Proof.

$$\begin{aligned}
CA - qAC &= \\
&= (z_{jk} - z_{j1} z_{11}^{-1} z_{1k})(z_{ik} - z_{i1} z_{11}^{-1} z_{1k}) - q(z_{ik} - z_{i1} z_{11}^{-1} z_{1k})(z_{jk} - z_{j1} z_{11}^{-1} z_{1k}) = \\
&\underbrace{z_{jk} z_{ik}}_{\boxed{1}} - \underbrace{z_{jk} z_{i1} z_{11}^{-1} z_{1k}}_{\boxed{2}} - \underbrace{z_{j1} z_{11}^{-1} z_{1k} z_{ik}}_{\boxed{3}} + \underbrace{z_{j1} z_{11}^{-1} z_{1k} z_{i1} z_{11}^{-1} z_{1k}}_{\boxed{4}} + \\
&- \underbrace{q z_{ik} z_{jk}}_{\boxed{1'}} + \underbrace{q z_{i1} z_{11}^{-1} z_{1k} z_{jk}}_{\boxed{2'}} + \underbrace{q z_{ik} z_{j1} z_{11}^{-1} z_{1k}}_{\boxed{3'}} - \underbrace{q z_{i1} z_{11}^{-1} z_{1k} z_{j1} z_{11}^{-1} z_{1k}}_{\boxed{4'}}
\end{aligned}$$

Now $\boxed{1} = \boxed{1'}$ and $\boxed{4} = \boxed{4'}$ and:

$$\begin{aligned}
\boxed{2} &= z_{11}^{-1} z_{1k} z_{i1} z_{jk} \\
\boxed{3} &= q^{-1} z_{11}^{-1} z_{1k} z_{ik} z_{j1} \\
\boxed{2} &= q^{-1} (z_{jk} z_{11}^{-1}) z_{i1} z_{1k} = q^{-1} z_{11}^{-1} z_{1k} (z_{jk} z_{i1}) - (q^{-1} - q) q^{-2} z_{11}^{-2} z_{1k}^2 z_{i1} z_{j1} = \\
&= z_{11}^{-1} z_{1k} z_{i1} z_{jk} - (q^{-1} - q) z_{11}^{-1} z_{1k} z_{ik} z_{j1} - (q^{-1} - q) q^{-2} z_{11}^{-2} z_{1k}^2 z_{i1} z_{j1} \\
\boxed{3} &= (z_{ik} z_{11}^{-1}) z_{j1} z_{1k} = q z_{11}^{-1} z_{1k} z_{ik} z_{j1} + (q^{-1} - q) q^{-2} z_{11}^{-2} z_{1k}^2 z_{i1} z_{j1}
\end{aligned}$$

The cancellations are obvious now. \square

Lemma 4 $AD - DA = (q^{-1} - q)BC$

Proof.

$$\begin{aligned}
AD - DA - (q^{-1} - q)BC &= \\
&= (z_{ik} - z_{i1} z_{11}^{-1} z_{1k})(z_{jl} - z_{j1} z_{11}^{-1} z_{1l}) - (z_{jl} - z_{j1} z_{11}^{-1} z_{1l})(z_{ik} - z_{i1} z_{11}^{-1} z_{1k}) - \\
&- (q^{-1} - q)(z_{il} - z_{i1} z_{11}^{-1} z_{1l})(z_{jk} - z_{j1} z_{11}^{-1} z_{1k}) = \\
&\underbrace{z_{ik} z_{jl}}_{\boxed{1}} - \underbrace{z_{i1} z_{11}^{-1} z_{1k} z_{jl}}_{\boxed{2}} - \underbrace{z_{ik} z_{j1} z_{11}^{-1} z_{1l}}_{\boxed{3}} + \underbrace{z_{i1} z_{11}^{-1} z_{1k} z_{j1} z_{11}^{-1} z_{1l}}_{\boxed{4}} + \\
&- \underbrace{z_{jl} z_{ik}}_{\boxed{1'}} + \underbrace{z_{jl} z_{i1} z_{11}^{-1} z_{1k}}_{\boxed{2'}} + \underbrace{z_{j1} z_{11}^{-1} z_{1l} z_{ik}}_{\boxed{3'}} - \underbrace{z_{j1} z_{11}^{-1} z_{1l} z_{i1} z_{11}^{-1} z_{1k}}_{\boxed{4'}} \\
&- (q^{-1} - q) \underbrace{(z_{il} z_{jk})}_{\boxed{1''}} - \underbrace{(z_{il} z_{j1} z_{11}^{-1} z_{1k})}_{\boxed{2''}} - \underbrace{(z_{i1} z_{11}^{-1} z_{1l} z_{jk})}_{\boxed{3''}} + \underbrace{(z_{i1} z_{11}^{-1} z_{1l} z_{j1} z_{11}^{-1} z_{1k})}_{\boxed{4''}}
\end{aligned}$$

Now $\boxed{1} - \boxed{1'} = \boxed{1''}$ and $\boxed{4} = q\boxed{4'}$, $\boxed{4'} = q^{-1}\boxed{4''}$ and we are left with:

$$\begin{aligned}
AD - DA - (q^{-1} - q)BC &= (\boxed{2} - (q^{-1} - q)\boxed{2'}) + (\boxed{2} - (q^{-1} - q)\boxed{3'}) + \boxed{3} - \boxed{3} = \\
&= (z_{jl}z_{i1} - (q^{-1} - q)z_{il}z_{j1})z_{11}^{-1}z_{1k} - z_{i1}z_{11}^{-1}(z_{1k}z_{jl} - (q^{-1} - q)z_{1l}z_{jk}) + \boxed{3} - \boxed{3} = \\
&= \underbrace{z_{i1}z_{jl}z_{11}^{-1}z_{1k}}_{\boxed{5}} - \underbrace{z_{i1}z_{11}^{-1}z_{jl}z_{1k}}_{\boxed{6}} + \boxed{3} - \boxed{3} \\
\boxed{5} &= z_{i1}(z_{jl}z_{11}^{-1})z_{1k} = z_{i1}z_{11}^{-1}z_{jl}z_{1k} + (q^{-1} - q)q^{-2}z_{i1}z_{11}^{-2}z_{1l}z_{j1}z_{1k} = \\
&= q^{-1}z_{11}^{-1}z_{i1}(z_{1k}z_{jl}) + (q^{-1} - q)q^{-3}z_{11}^{-2}z_{1k}z_{1l}z_{j1}z_{i1} = \\
&= q^{-1}z_{11}^{-1}z_{i1}z_{jl}z_{1k} - (q^{-1} - q)q^{-1}z_{11}^{-1}z_{1l}z_{i1}z_{jk} + (q^{-1} - q)q^{-3}z_{11}^{-2}z_{1k}z_{1l}z_{j1}z_{i1} \\
\boxed{6} &= q^{-1}z_{11}^{-1}z_{i1}(z_{jl}z_{1k}) = q^{-1}z_{11}^{-1}z_{i1}z_{1k}z_{jl} - (q^{-1} - q)q^{-1}z_{11}^{-1}z_{1l}z_{i1}z_{jk} \\
\boxed{3} &= q^{-1}(z_{ik}z_{11}^{-1})z_{j1}z_{1l} = q^{-1}z_{11}^{-1}z_{ik}z_{1l}z_{j1} - (q^{-1} - q)q^{-2}z_{11}^{-2}z_{1k}z_{i1}z_{1l}z_{j1} = \\
&= q^{-1}z_{11}^{-1}z_{1l}z_{ik}z_{j1} - (q^{-1} - q)q^{-3}z_{11}^{-2}z_{1k}z_{1l}z_{i1}z_{j1} \\
\boxed{3} &= q^{-1}z_{11}^{-1}z_{1l}z_{ik}z_{j1}
\end{aligned}$$

This also ends the proof of the Lemma and the proof of the Theorem. \square

Remark 2 A warning is in order, it is not true that by any elementary transormation a quantum matrix would change to a new quantum matrix. We proved that this is only in the case when one performs the usual elementary transformations such that all elements in the first column except just one are set to zero, this is the matrix we denoed by Z' . Of course, now this can be iterated several times in order to obtain an upper triangular matrix.

Remark 3 The case $n = 3$ The quantum determinant is:

$$\det_q(Z_3) = z_{11}z_{22}z_{33} - q^{-1}z_{12}z_{21}z_{33} - q^{-1}z_{11}z_{23}z_{32} + q^{-2}z_{12}z_{23}z_{31} + q^{-2}z_{13}z_{21}z_{23} - q^{-3}z_{13}z_{22}z_{31}.$$

One can prove through laborious calculations (see [PH]) that the coset of the $\det_q(Z_3)$ mod commutators equals the Dieudonné determinant of the 3 by 3 quantum matrix Z_3 .

Remark 4 At this point, induction would work if the quantum determinant were multiplicative. This is not the case. For instance for the quantum matrix Z corresponding to the parameter q , the square matrix Z^2 is a quantum matrix for the parameter q^2 , this is a quite astonishing fact that calls for explanation! Due to the fact that $\det_q(Z)$ is a group-like element, one has a weak multiplicative property namely if A and B are quantum matrices such that their elements commute pairwise, then the product AB is again a quantum matrix and $\det_q(AB) = \det_q(A)\det_q(B)$.

But in the case when we perform elementary row operations on Z , say A is a product of transvections and $B = Z$, the components of these matrices do not commute, so the above-mentioned result does not apply.

Meanwhile we can prove directly the following result which is all we need:

Theorem 4 (Row reducing the quantum determinant) The quantum determinants of the matrix Z' obtained by row-reducing a quantum matrices Z as in formulas (4) and (5) is equal to the quantum determiant of the original quantum matrix:

$$|Z|_q = |Z'|_q \quad (6)$$

Proof. Let $E' = k < z_{ij}, z_{11}^{-1} >$, $R' = E' \otimes_k B$, and $\xi'_i = \sum_j z_{ij} \xi_j$, and $\xi''_i = \sum_j z'_{ij} \xi_j$.

In R' we have:

$$\xi'_1 \xi'_2 \cdots \xi'_n = \det_q(Z) \xi_1 \xi_2 \cdots \xi_n \quad \text{and} \quad \xi''_1 \xi''_2 \cdots \xi''_n = \det_q(Z') \xi_1 \xi_2 \cdots \xi_n.$$

If we denote $t_i := -z_{i1} z_{11}^{-1}$ then $\xi''_i = \xi'_i + t_i \cdot \xi'_1$ for any $i > 1$ we may write:

$$\xi''_1 \xi''_2 \cdots \xi''_n = \xi'_1 (\xi'_2 + t_2 \xi'_1) \cdots (\xi'_n + t_n \xi'_1).$$

By induction we prove now:

$$\xi'_1 (\xi'_2 + t_2 \xi'_1) \cdots (\xi'_n + t_n \xi'_1) = \xi'_1 \xi'_2 \cdots \xi'_{i-1} (\xi'_i + t_i \xi'_1) \cdots (\xi'_n + t_n \xi'_1) \quad (7)$$

Indeed:

$$\begin{aligned} & \xi'_1 \xi'_2 \cdots \xi'_{i-1} (\xi'_i + t_i \xi'_1) \cdots (\xi'_n + t_n \xi'_1) = \\ & \xi'_1 \xi'_2 \cdots \xi'_{i-1} \xi'_i (\xi'_{i+1} + t_{i+1} \xi'_1) \cdots (\xi'_n + t_n \xi'_1) + \xi'_1 \xi'_2 \cdots \xi'_{i-1} \cdot t_i \xi'_1 (\xi'_{i+1} + t_{i+1} \xi'_1) \cdots (\xi'_n + t_n \xi'_1) = \\ & \xi'_1 \xi'_2 \cdots \xi'_{i-1} \xi'_i (\xi'_{i+1} + t_{i+1} \xi'_1) \cdots (\xi'_n + t_n \xi'_1) + \xi'_1 \xi'_2 \cdots \xi'_{i-1} \cdot \xi'_1 q t_i (\xi'_{i+1} + t_{i+1} \xi'_1) \cdots (\xi'_n + t_n \xi'_1) \end{aligned}$$

because a direct calculation shows that $t_i \xi'_1 = q \xi'_1 t_i$. But using the fact that ξ'_i are points of the quantum Grassmannian plane, by Proposition 2 i.e., $\xi_i'^2 = 0$ for all i , and $\xi'_i \xi'_j = q^{-1} \xi'_j \xi'_i$ for all $i < j$, it follows that $\xi'_1 \xi'_2 \cdots \xi'_{i-1} \cdot \xi'_1 = 0$ so the second term vanishes. We shall later need a multiparameter version of this result, you may see it holds as well. Eventually for $i = n$ we get:

$$\xi''_1 \xi''_2 \cdots \xi''_n = \xi'_1 (\xi'_2 + t_2 \xi'_1) \cdots (\xi'_n + t_n \xi'_1) = \xi'_1 \xi'_2 \cdots \xi'_n$$

therefore

$$\xi'_1 \xi'_2 \cdots \xi'_n = \det_q(Z) \xi_1 \xi_2 \cdots \xi_n = \xi''_1 \xi''_2 \cdots \xi''_n = \det_q(Z') \xi_1 \xi_2 \cdots \xi_n$$

hence: $\det_q(Z) = \det_q(Z')$. □

We can now prove

Corollary (Quantum determinant and Dieudonné determinant) *Let Z be a quantum matrix i.e., its elements satisfy the relations Rel_q . Then*

$$\pi(\det_q(Z)) = \det_D(Z).$$

Proof. Using the result above above $|Z|_q = |Z'|_q$. Now by column expansion $|Z'|_q = z_{11} |Z''|_q$. Because Z' hence also Z'' are quantum matrices and the dimension of Z'' is $n - 1$, by induction $\pi(|Z''|_q) = \det_D(Z'')$. On the other hand by Theorem 2 $\det_D(Z) = \pi(z_{11}) \det_D(Z'')$. □

5 Row-reducing in the multiparametric case

In this section we look at the quantum determinant for the multiparameter quantum linear group over a field k . We shall use the notation and results from [AST].

If $p = (p_{ij})$ is such that $p_{ij}p_{ji} = 1$ and $p_{ii} = 1$, we call such a matrix p a multiplicatively antisymmetric matrix. If $\lambda, p_{ij} \in k^*$ define the multiparameter quantum linear group to be a k -algebra with generators u_{ij} , $M_{p,\lambda}(n) = k \langle u_{11}, u_{12}, \dots, u_{nn} \rangle$ subject to the following *twisted quantum relations*:

$$TWRel_{p,\lambda} \left\{ \begin{array}{l} \text{Row relations: } u_{il} u_{ik} = \frac{1}{p_{lk}} u_{ik} u_{il} \text{ for all } k < l \\ \text{Column relations: } u_{jk} u_{ik} = \lambda p_{ji} u_{ik} u_{jk} \text{ for all } i < j \\ \text{Secondary diagonal relations: } u_{jk} u_{il} = \frac{\lambda p_{ji}}{p_{kl}} u_{il} u_{jk} \text{ for all } i < j, k < l \\ \text{Main diagonal relations: } u_{jl} u_{ik} = \frac{p_{ji}}{p_{kl}} u_{ik} u_{jl} + (\lambda - 1) p_{ji} u_{il} u_{jk} \text{ if } i < j, k < l. \end{array} \right.$$

We denote by U the matrix (u_{ij}) , with u_{ij} satisfying the above twisted quantum relations $TWRel_{p,\lambda}$, and let $q_{ji} := \lambda p_{ji}$ for all $i < j$, $q_{ii} := 1$ and $q_{ji} := \lambda^{-1} p_{ij}$ for $i > j$, and $I = \{1, \dots, n\}$.

The comultiplication on $M_{p,\lambda}(n)$ is dual to matrix multiplication: $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$, and the counit is defined by $\epsilon(z_{ij}) = \delta_{ij}$, both extended multiplicatively to all $M_{p,\lambda}(n)$. Then $M_{p,\lambda}(n)$ has a bialgebra structure. $M_{p,\lambda}(n)$ can be obtained by using the Manin quadratic algebra construction for the following two quadratic algebras:

- $A = k \langle x_1, \dots, x_n \rangle$ subject to relations : $x_j x_i = q_{ji} x_i x_j$ for all $i, j \in I$
- $B = k \langle y_1, \dots, y_n \rangle$ subject to relations : $y_j y_i = p_{ji} y_i y_j$ for all $i, j \in I$

We proceed as in [AST] to define the determinant, namely we consider the corresponding Grassmannian algebras $A^!$ and $B^!$

- $A^! = k \langle \xi_1, \dots, \xi_n \rangle$ subject to relations: $\xi_j \xi_i = -q_{ij} \xi_i \xi_j$ and $\xi_i^2 = 0, \forall i, j \in I$
- $B^! = k \langle \eta_1, \dots, \eta_n \rangle$ subject to relations: $\eta_j \eta_i = -p_{ji} \eta_i \eta_j$ and $\eta_i^2 = 0, \forall i, j \in I$

For any subset J of I write $\xi_J = \prod_{j \in J} \xi_j$ and $\eta_J = \prod_{j \in J} \eta_j$ the product is taken in increasing order.

Let $H = H(p, \lambda) = M_{p,\lambda}(n)$, and consider the coactions:

$$\delta : A^! \longrightarrow A^! \otimes H \text{ and } \delta : B^! \longrightarrow H \otimes B^! \text{ they are both algebra homomorphisms.}$$

If $\delta(\xi_j) = \sum_k u_{jk} \otimes \xi_k$ and $\delta(\eta_j) = \sum_k u_{jk} \otimes \eta_k$ then it follows that

$$\delta(\xi_K) = \sum_J \xi_J \otimes U_{JK} \text{ and } \delta(\eta_K) = \sum_J U_{KJ} \otimes \eta_J$$

where, as in [AST], Lemma 1 : $[J, K]$ is the set of bijective mappings $\theta : J \xrightarrow{\sim} K$ for any subsets J, K of I and by definition

$$U_{J,K} := \sum_{\theta \in [J,K]} \sigma(p, \theta) \prod_{j \in J} u_{j, \theta j} = \sum_{\theta \in [J,K]} \sigma(q^{-1}, \theta) \prod_{k \in K} u_{\theta k, k}$$

with the products taken in increasing order and $\sigma(p, \theta) = \prod_{j < j', \theta_j > \theta_{j'}} (-p_{\theta_j, \theta_{j'}})$.

In particular for $J = K = I$ we get $\det_{p, \lambda}(U) = U_{I, I}$.

Definition *The multiparameter quantum determinant is:*

$$\det_{p, \lambda}(U) = U_{I, I} = \sum_{\theta \in S_n} \sigma(p, \theta) u_{1, \theta_1} \cdots u_{n, \theta_n} \quad \text{where } \sigma(p, \theta) = \prod_{j < j', \theta_j > \theta_{j'}} (-p_{\theta_j, \theta_{j'}}) \quad (8)$$

As we did before for the quantum matrices, if $H = k < u_{ij} >, R = H \otimes_k B$, then a reformulation of this definition is that in R we have:

Proposition 6 *If $\eta'_i = \sum_j u_{ij} \eta_j$ then $\eta'_1 \eta'_2 \cdots \eta'_n = \det_{p, \lambda}(U) \eta_1 \eta_2 \cdots \eta_n$.*

In a similar way we have formulas for rows and columns expansion. Denote by U_{ji} the determinant $\det_{p, \lambda}$ of the $(n-1)$ by $(n-1)$ matrix obtained by removing the j -th row and the i -th column of U .

Let us denote $\beta_j := \prod_{m=j+1}^n (-q_{jm})$ and $\gamma_j := \prod_{m=1}^{j-1} (-p_{jm})$.

The following is proved in [AST] Thm 3, (21).

Proposition 7 *If U is a quantum multiparameter matrix then*

- 1) *The element $\det_{p, \lambda}(U)$ is normalizing (but it is not central)*
- 2) *The following row and column expansion formulas hold:*

$$\det_{p, \lambda}(U) = \sum_j \frac{\beta_j}{\beta_k} U_{jk} u_{jk} = \sum_k \frac{\gamma_k}{\gamma_j} u_{jk} U_{jk}. \quad (9)$$

Another important fact established in [AST] that we need to use is the fact that $M_{p, \lambda}(n)$ can be obtained by twisting the multiplication in $M_q(n)$ by a cocycle associated to p .

First note that for $\lambda = q^2$ and $p'_{ij} = q$, for $i < j$, $p'_{ij} = q^{-1}$ for $i > j$ and $p'_{ii} = 1$ the corresponding multiparametric quantum matrix ring $M_{p', q^2}(n)$ is just $M_q(n)$.

Now let G be an abelian group isomorphic to the product of n copies of Z . Using the multiplicative notation and denoting by t_i the generator of the i -th copy we may write

$$G \approx \langle t_1 \rangle \times \langle t_2 \rangle \times \cdots \times \langle t_n \rangle. \quad (10)$$

We give $M_{p, \lambda}(n)$ a grading by letting u_{ij} have left degree equal to t_i and right degree equal to t_j and extend this multiplicatively. Then if we remark that $M_{p, \lambda}(n)$ is an iterated Ore extension, and if we let D be its total field of fractions, then u_{ij}^{-1} , an element in D , has left degree t_i^{-1} and right degree t_j^{-1} .

For an arbitrary antisymmetric matrix $p = (p_{ij})$ such that $p_{ij}p_{ji} = 1$ and $p_{ii} = 1$ define the 2-cocycle on G by:

$$c_p(\prod t_i^{m_i}, \prod t_j^{n_j}) := \prod_{i < j} p_{ij}^{m_i n_j},$$

in fact this is the unique bimultiplicative function such that:

$$c_p(t_i, t_j) = p_{ij} \text{ if } i < j \text{ and } c_p(t_i, t_j) = 1 \text{ if } i \geq j.$$

Conversely for any 2-cocycle c on G we associate the following matrix $r(c)$, which has the property that p is antisymmetric:

$$r_{ij}(t_i, t_j) = \frac{c(t_i, t_j)}{c(t_j, t_i)}.$$

Proposition 1 in [AST] shows that the correspondences defined above define a bijection from $H^2(G, k^*)$ to the set of multiplicatively antisymmetric matrices p , i.e., with the property $p_{ij}p_{ji} = 1$ and $p_{ii} = 1$

If A is a $G \times G$ -graded algebra, having both a left and a right grading we define a new multiplication " \circ " on A called the multiplication twisted by c_p^{-1} on the left and by c_p on the right by the formula:

$$a \circ b = c_p^{-1}(t_i, t_j) c_p(t_k, t_l) a \cdot b \quad (11)$$

where a has left degree t_i and right degree t_k , b has left degree t_j and right degree t_l .

We write ${}_{c^{-1}}A_c$ for the algebra A with the twisted multiplication \circ .

We need also the following result established in [AST] Thm 4:

Proposition 8 *If we twist $H(p, \lambda) = M_{p, \lambda}(n)$ simultaneously by c_p^{-1} on the left and by c_p on the right we obtain*

$${}_{c^{-1}}M_{p, \lambda}(n)_c = M_{r(c)p, \lambda}(n) \quad \text{or} \quad {}_{c^{-1}}H(p, \lambda)_c = H(r(c)p, \lambda).$$

In particular $M_{p, \lambda}(n)$ can be obtained by twisting $M_q(n) = H(p', \lambda)$ (for $\lambda = q^2$ and $p'_{ij} = q$, for $i < j$, $p'_{ij} = q^{-1}$ for $i > j$ and $p'_{ii} = 1$) by the cocycle defined above c_p . In this particular case we shall not use \circ any more for the multiplication but simply juxtaposition.

We are now ready now to look at the determinant of the matrix U . First we look at

Example 4 (The multiparametric quantum determinant for $n = 2$)

The multiparametric quantum determinant is:

$$\begin{vmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{vmatrix}_{p, \lambda} = u_{11}u_{22} - p_{21}u_{12}u_{21}$$

The Dieudonné determinant is

$$\det_D(U) = u_{11}(u_{22} - u_{21}u_{11}^{-1}u_{12}).$$

Now use the defining relations: $u_{21}u_{11}^{-1} = \frac{1}{\lambda p_{21}}u_{11}^{-1}u_{21}$ and $u_{21}u_{12} = \frac{\lambda p_{21}}{p_{12}}u_{12}u_{21}$.

Use also $\frac{1}{p_{12}} = p_{21}$ and this will establish $\pi(\det_{p,\lambda}(U)) = \det_D(U)$ for $n = 2$.

We may now prove that if U is a multiparametric quantum matrix satisfying the twisted relations $TWRel_{p,\lambda}$ then for any n we have $\pi(\det_{p,\lambda}(U)) = \det_D(U)$.

Let us define the matrix $U' = (u'_{ij})$ by

$$u'_{1j} = u_{1j} \text{ for any } j, u'_{i1} = 0, \text{ for } i = 2, \dots, n, \quad (12)$$

$$u'_{ij} = u_{ij} - u_{i1}u_{11}^{-1}u_{1j} \text{ for } i = 2, \dots, n \text{ and } j = 2, \dots, n. \quad (13)$$

U' is obtained from U by clearing up all the positions of the first column except the first one by use of elementary row transformations.

Let U'' be the $(n-1) \times (n-1)$ matrix obtained by deleting the first row and the first column of U' . Just like before we can establish the following

Theorem 5 (Row-reducing the multiparametric quantum matrix) *The matrix U' is a multiparametric quantum matrix, i.e., its elements satisfy the relations $TWRel_{p,\lambda}$ defining $M_{p,\lambda}(n)$*

Proof. The proof relies on the following fact: each relation in $TWRel_{p,\lambda}$ is obtained by twisting the corresponding relation in Rel_q (the relations of $M_q(n)$). We use the identification $z_{ij} \longleftrightarrow u_{ij}$ and keep in mind that the product of the u_{ij} is obtained by twisting by c_p^{-1} on the left and by c_p on the right the multiplication of the corresponding z_{ij} , so we get a factor like $c_p(t_i, t_j) c_p^{-1}(t_k, t_l)$ when twisting the product $z_{ik} \cdot z_{jl}$.

Now a direct calculation shows that $u_{i1}u_{11}^{-1}u_{1j}$ comes from the corresponding product $z_{i1}z_{11}^{-1}z_{1j}$ changed by the factor $c_p(t_i, t_1^{-1}) c_p^{-1}(t_1, t_1^{-1}) c_p(t_1 t_1^{-1}, t_1^{-1}) c_p^{-1}(e, t_j)$. This factor is equal to 1 because c_p is bimultiplicative by its definition. It means the product is not changed by the twist. It has the same left and same right degree as u_{ij} .

Therefore u'_{ij} has left degree t_i and right degree t_j , in fact the same left and same right degree as u_{ij} . Then by twisting (by c_p^{-1} on the left and by c_p on the right) the products in the relations Rel_q satisfied by z'_{ij} we get that the relations $TWRel_{p,\lambda}$ are satisfied by u'_{ij} , because the degrees are the same for like for z_{ij} and z'_{ij} . We can see now that u'_{ij} satisfy exactly the twisted relations $TWRel_{p,\lambda}$. This establishes the theorem. \square

Corollary (Multiparametric quantum determinant and Dieudonné determinant) *Let U be a multiparametric quantum matrix satisfying the relations $TWRel_{p,\lambda}$. Then for any n we have*

$$\pi(\det_{p,\lambda}(U)) = \det_D(U).$$

Proof. We use induction exactly as in the proof of the quantum determinant.

The proof for $|Z|_q = |Z'|_q$ works exactly in the same way, establish that $|U|_{p,\lambda} = |U'|_{p,\lambda}$ and by column expansion $|U'|_q = u_{11}|U''|_q$.

Because U' hence also U'' are multiparametric quantum matrices and the dimension of U'' is $n - 1$, by induction $\pi(|U''|_q) = \det_D(U'')$. On the other hand by Theorem 2, $\det_D(U) = \pi(u_{11})\det_D(U'')$ which proves our result. \square

Remark 5 *We proved the Theorems 3 and 5 and their corollaries for the generic matrices Z and U . In this case we do not have to worry about the existence of z_{11}^{-1} or u_{11}^{-1} . For an arbitrary quantum matrix or quantum multiparameter matrix we may now use specialization.*

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